FORMAL SOLUTION OF SCALAR RICCATI EQUATION

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ABSTRACT. Formal solution of Riccati equation is received.

1. Scalar Riccati equation

General Riccati equation[1]

$$\frac{dy(x)}{dx} = a(x)y^2(x) + b(x)y(x) + c(x) \tag{1.1}$$

has solution in a form of infinite procedure

$$y(x) = \frac{1}{2a}(-b\pm\sqrt{b^2 - 4a(c - \frac{d}{dx}(\frac{1}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{d}{dx}(\frac{1}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{d}{dx}(...)))))))))} \tag{1.2}$$

Direct substitution of $y(x)$ from Eq. 1.2 to Eq. 1.1 transforms last to identity:

$$\begin{aligned} ay^2 + by + c &= \\ \frac{1}{4a}(b^2 \mp 2b\sqrt{b^2 - 4a(c - \frac{dy}{dx})} + b^2 - 4a(c - \frac{dy}{dx})) + \frac{b}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{dy}{dx})}) + c &= \frac{dy}{dx} \end{aligned}$$

The procedure described above can be expressed in terms of iterations:

$$y_0(x) = \frac{1}{2a(x)}[-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x)}]$$

$$y_{n+1}(x) = \frac{1}{2a(x)}[-b(x) \pm \sqrt{b^2(x) - 4a(x)(c(x) - \frac{dy_n}{dx})}], n \geq 0$$

Example of this approach applied to the receiving of approximate solutions of Airy equation[2] in the following section.

2. Airy equation

Solutions of the Airy equation

$$\frac{d^2y(x)}{dx^2} + xy(x) = 0 \tag{2.1}$$

can be looked in the form

$$y(x) = \exp(i\int^x q(x)dx) \tag{2.2}$$

and for $q(x)$ it becomes Riccati equation

$$q^2(x) - x = i\frac{dq(x)}{dx} \tag{2.3}$$

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According to Eq. 1.2 formal solution of this equation has the form

$$q(x) = \pm\sqrt{x \pm i\frac{d}{dx}\sqrt{x \pm i\frac{d}{dx}\sqrt{x...}}} \tag{2.4}$$

If to take first two iterations

$$\begin{aligned} q_0(x) &= \pm x^{0.5} \\ q_1(x) &= \pm\sqrt{x \pm \frac{i}{2}x^{-0.5}} = \pm x^{-0.25}\sqrt{x^{1.5} \pm \frac{i}{2}} \end{aligned}$$

Then

$$\begin{aligned} I_0^{(1,2)}(x) &= \int^x q_0(t)dt = \pm\frac{2}{3}x^{\frac{3}{2}} \\ I_1^{(1,2)}(x) &= \int^x q_1(t)dt = \pm\frac{2}{3}x^{\frac{3}{4}}\sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}} + \frac{i}{3}\ln(x^{\frac{3}{4}} + \sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}}) \\ I_1^{(1,2)}(0) &= \frac{i}{3}\ln(\sqrt{\pm\frac{i}{2}}) = \frac{i}{6}\ln(\pm\frac{i}{2}) = \frac{i}{6}(\pm i\frac{\pi}{2} - \ln 2) = \mp\frac{\pi}{12} - i\frac{\ln 2}{6} \end{aligned}$$

According to [2] first two terms of asymptotic expansion for Airy functions when $\Re x \to \infty, \Im x = 0$ are:

$$\begin{aligned} Ai(x) &\sim \pi^{-\frac{1}{4}}x^{-\frac{1}{4}}\sin(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) \\ &= \frac{1}{i\pi^{\frac{1}{4}}}(\exp(i(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) - \frac{1}{4}\ln x - \ln 2) - \exp(-i(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) - \frac{1}{4}\ln x - \ln 2)) \\ Bi(x) &\sim \pi^{-\frac{1}{4}}x^{-\frac{1}{4}}\cos(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) \\ &= \frac{1}{\pi^{\frac{1}{4}}}(\exp(i(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) - \frac{1}{4}\ln x - \ln 2) + \exp(-i(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) - \frac{1}{4}\ln x - \ln 2)) \end{aligned}$$

When $\Re x \to -\infty, \Im x = 0$

$$\begin{aligned} Ai(x) &\sim \pi^{-\frac{1}{2}}\exp(-\frac{2}{3}(-x)^{\frac{3}{2}} - \frac{1}{4}\ln(-x)) \\ Bi(x) &\sim \pi^{-\frac{1}{4}}\exp(\frac{2}{3}(-x)^{\frac{3}{2}} - \frac{1}{4}\ln(-x)) \end{aligned}$$

When $\Re x \to \infty, \Im x = 0$

$$iI_1^{(1,2)}(x) \sim \pm i\frac{2}{3}x^{\frac{3}{2}} - \frac{\ln 2}{3} - \frac{1}{4}\ln x \tag{2.5}$$

When $\Re x \to -\infty, \Im x = 0$

$$iI_1^{(1,2)}(x) \sim \mp\frac{2}{3}(-x)^{\frac{3}{2}} - \frac{\ln 2}{3} - \frac{\pi}{3} - \frac{1}{4}\ln(-x) \tag{2.6}$$

Comparing Eq. 2.5-2.6 with asymptotic expansions we receive approximate expressions for Airy functions for $\Re x \geq 0, \Im x = 0$:

$$Ai(x) \sim \frac{1}{i\pi^{\frac{1}{4}}2^{\frac{2}{3}}}(\exp(i(I_1^{(1)}(x) + \frac{\pi}{4})) - \exp(i(I_1^{(2)}(x) - \frac{\pi}{4}))) \tag{2.7}$$

$$Bi(x) \sim \frac{1}{\pi^{\frac{1}{4}}2^{\frac{2}{3}}}(\exp(i(I_1^{(1)}(x) + \frac{\pi}{4})) + \exp(i(I_2^{(1)}(x) - \frac{\pi}{4}))) \tag{2.8}$$

and for $\Re x \leq 0, \Im x = 0$

$$Ai(x) \sim \sin\frac{\pi}{6}\frac{2^{\frac{1}{3}}}{\pi^{\frac{1}{4}}}\Re\exp(i(I_1^{(1)}(x)+\frac{\pi}{12})) \tag{2.9}$$

$$Bi(x) \sim \cos\frac{\pi}{6}\frac{2^{\frac{1}{6}}}{\pi^{\frac{1}{4}}}\Re\exp(-i(I_1^{(1)}(x)+\frac{\pi}{12})) \tag{2.10}$$

(2.11)

At Fig.1 approximate values of $Ai(x), Bi(x)$ are displayed. They have discontinuity of first derivative at $x = 0$, while traditional asymptotic expressions are singular at $x = 0$.

3. Appendix A

For computation of integral

$$I_1(x) = \int^x q_1(t)dt = \pm\int^x \sqrt{t \pm \frac{i}{2}t^{-0.5}}dt$$

we introduce new independent variable

$$y = t^{\frac{3}{4}}, t = y^{\frac{4}{3}}, dt = \frac{4}{3}y^{\frac{1}{3}}dy$$

and

$$I_1(x) = \pm\frac{4}{3}\int^x \sqrt{y^{\frac{4}{3}} \pm \frac{i}{2}y^{-\frac{2}{3}}}y^{\frac{1}{3}}dy = \pm\frac{4}{3}\int^x \sqrt{y^2 \pm \frac{i}{2}}dy =$$
$$\pm\frac{4}{3}(\frac{1}{2}y\sqrt{y^2 \pm \frac{i}{2}} \pm \frac{i}{4}\ln(y + \sqrt{y^2 \pm \frac{i}{2}}))|^x =$$
$$\pm\frac{2}{3}(x^{\frac{3}{4}}\sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}} \pm \frac{i}{2}\ln(x^{\frac{3}{4}} + \sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}}))$$

References

1.Agnew,R.P,:Differential Equations, McGraw-Hill, NY,1960

2.Abramowitz, M.,Stegun I.A.,Handbook of mathematical functions, 1964

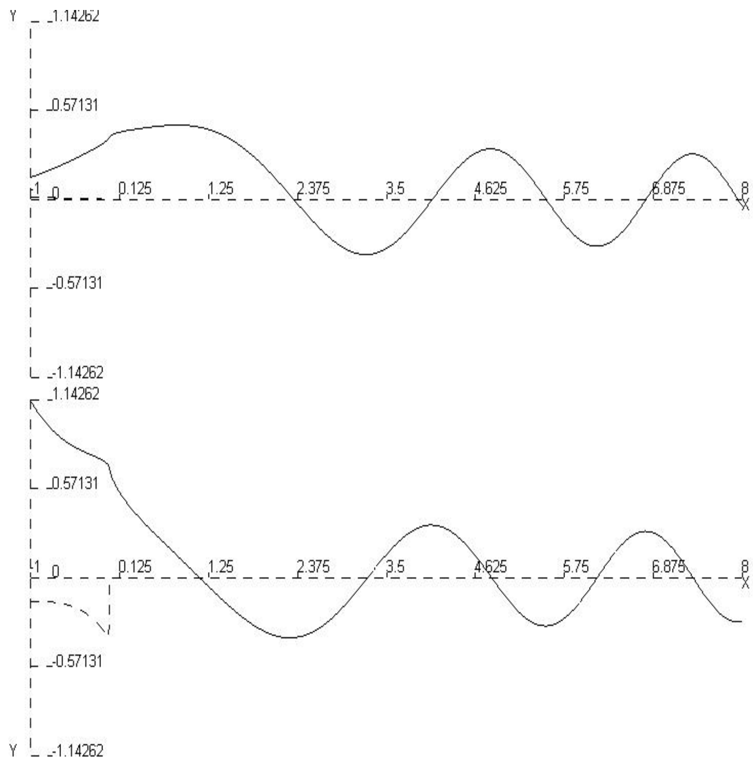

FIGURE 1. Approximate solutions of Airy equation