

# How many equators have 5-dimensional sphere?

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## 1 3-dimensional sphere

Equation of the *body* of the sphere in 3-D Euclidean space  $(x_1, x_2, x_3)$  is [1]:

$$x_1^2 + x_2^2 + x_3^2 \leq R^2 \quad (1)$$

It is a 3-D object. A *surface* of 3-D sphere

$$x_1^2 + x_2^2 + x_3^2 = R^2 \quad (2)$$

is a 2-dimensional, since  $x_1, x_2, x_3$  are tied with equation (2), so only two of them are independent. It especially can be seen from parametric representation of sphere surface as

$$x_1 = R \cos \alpha_0 \cos \alpha_1 \quad (3)$$

$$x_2 = R \cos \alpha_0 \sin \alpha_1 \quad (4)$$

$$x_3 = R \sin \alpha_0 \quad (5)$$

$$-\frac{\pi}{2} < \alpha_0 < \frac{\pi}{2}, 0 \leq \alpha_1 < 2\pi, \quad (6)$$

where  $\alpha_0$  is, of course, a *latitude*, and  $\alpha_1$  is a *longitude*. If  $\alpha_0 = 0$  then a circle in plane  $(x_1, x_2)$

$$x_1 = R \cos \alpha_1$$

$$x_2 = R \sin \alpha_1$$

$$x_3 = 0$$

$$0 \leq \alpha_1 < 2\pi,$$

is a 1-dimensional *equator* of 3-dimensional sphere. When  $\alpha_0 = -\frac{\pi}{2}$

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = -R$$

we have *south pole* and When  $\alpha_0 = \frac{\pi}{2}$

$$\begin{aligned}x_1 &= 0 \\x_2 &= 0 \\x_3 &= R\end{aligned}$$

it is, of course *a north pole*. When, at last,  $\alpha_1 = 0$ , a big half circle

$$\begin{aligned}x_1 &= R \cos \alpha_0 \\x_2 &= 0 \\x_3 &= R \sin \alpha_0 \\-\frac{\pi}{2} &< \alpha_0 < \frac{\pi}{2}\end{aligned}$$

in the plane  $(x_1, x_3)$  is *a zero meridian*, named Greenwich meridian on our earth. Intersection of the meridian with equator ( $\alpha_0 = 0, \alpha_1 = 0$ ) is a single point

$$x_1 = R, x_2 = x_3 = 0$$

Last expression for meridian can be obtained by rotation of one point around axis  $x_2$ :

$$\begin{vmatrix} R \cos \alpha_0 \\ 0 \\ R \sin \alpha_0 \end{vmatrix} = \begin{vmatrix} \cos \alpha_0, & 0, & -\sin \alpha_0 \\ 0, & 1, & 0 \\ \sin \alpha_0, & 0, & \cos \alpha_0 \end{vmatrix} \begin{vmatrix} R \\ 0 \\ 0 \end{vmatrix}$$

Finally equation of 3-D sphere can be obtained by rotation of single point around axis  $x_2$  to receive zero meridian and by rotation of zero meridian around axis  $x_3$ :

$$\begin{vmatrix} \cos \alpha_1, & -\sin \alpha_1, & 0 \\ \sin \alpha_1, & \cos \alpha_1, & 0 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} \cos \alpha_0, & 0, & -\sin \alpha_0 \\ 0, & 1, & 0 \\ \sin \alpha_0, & 0, & \cos \alpha_0 \end{vmatrix} \begin{vmatrix} R \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} R \cos \alpha_0 \cos \alpha_1 \\ R \cos \alpha_0 \sin \alpha_1 \\ R \sin \alpha_0 \end{vmatrix} \quad (7)$$

Result is -to receive a 2-D surface of 3-D sphere we must take a single point and perform 2 rotations, around 2 axes. Result is identical to initial Eq.(3-5).

Let us ascend at one more space dimension.

## 2 4-dimensional sphere

Equation of the *body* of the sphere in 4-D Euclidean space  $(x_1, x_2, x_3, x_4)$  is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2 \quad (8)$$

It is a 4-D object. A *surface* of 4-D sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \quad (9)$$

is a 3-dimensional, since  $x_1, x_2, x_3, x_4$  are tied by equation (9), so only three of them are independent. It especially can be seen from parametric representation of sphere surface as

$$x_1 = R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \quad (10)$$

$$x_2 = R \cos \alpha_0 \cos \alpha_1 \sin \alpha_2 \quad (11)$$

$$x_3 = R \cos \alpha_0 \sin \alpha_1 \quad (12)$$

$$x_4 = R \sin \alpha_0 \quad (13)$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_1 < \frac{\pi}{2}, 0 \leq \alpha_2 < 2\pi \quad (14)$$

where  $\alpha_0, \alpha_1$  are *two latitudes* and  $\alpha_2$  is *a longitude*. We can see all 2-D projections of 4-D sphere at Figure 1. Number of projections it is the number of combinations from four elements by two:

$$C_4^2 = \frac{4 \times 3}{2} = 6$$

Numbers in top-left and right-bottom corners of projections correspond to the coordinate axes  $x_1, x_2, x_3, x_4$ .

If  $\alpha_0 = 0$  then

$$x_1 = R \cos \alpha_1 \cos \alpha_2$$

$$x_2 = R \cos \alpha_1 \sin \alpha_2$$

$$x_3 = R \sin \alpha_1$$

$$x_4 = 0$$

$$-\frac{\pi}{2} < \alpha_1 < \frac{\pi}{2}, 0 \leq \alpha_2 < 2\pi$$

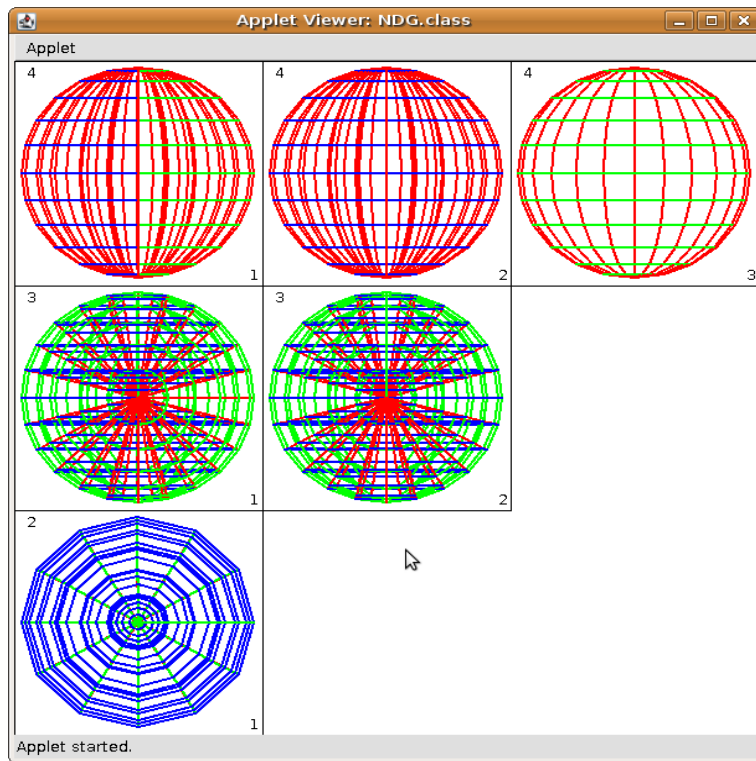


Figure 1: 4D Sphere. Red lines correspond to the change of  $\alpha_0$ , green lines correspond to the change of  $\alpha_1$ , blue lines correspond to the change of  $\alpha_2$ ,

is a 2-dimensional surface of 3-dimensional sphere, first *equator*, of 4-dimensional sphere in *hyperplane*  $(x_1, x_2, x_3)$ . We can see this object of Figure 3. Term

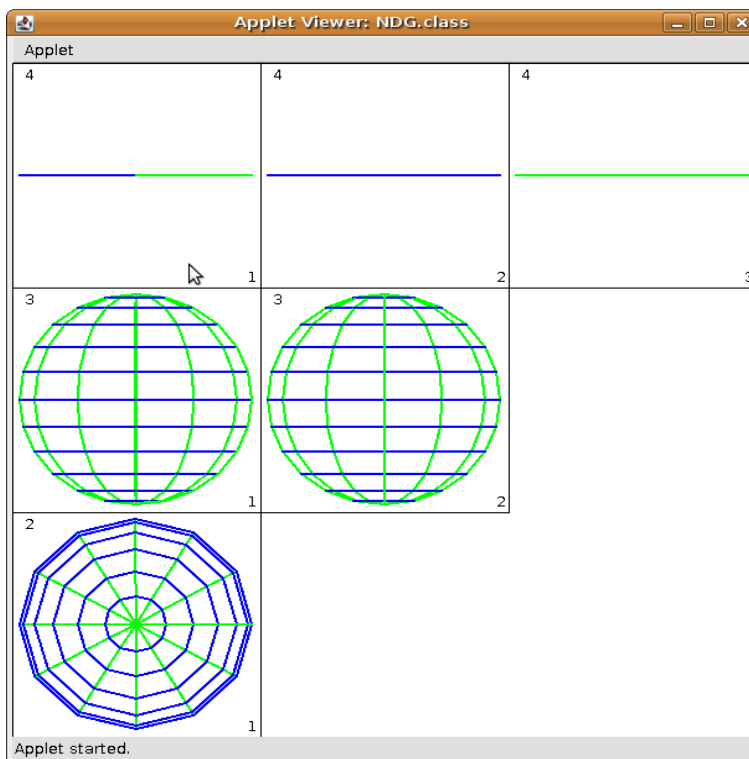


Figure 2: First equator of 4D Sphere. Red lines are absent, since  $\alpha_0 = 0$ , green lines correspond to the change of  $\alpha_1$ , blue lines correspond to the change of  $\alpha_2$ , The equator is a 3-D sphere in hyperplane  $(x_1, x_2, x_3)$ . Coordinate  $x_4 = 0$

*hyperplane* means that dimension of this plane is lower than dimension of all space under consideration. Point on 1-D line is 0-D hyperplane, with respect to that line and so on.

If  $\alpha_1 = 0$  then

$$x_1 = R \cos \alpha_0 \cos \alpha_2$$

$$x_2 = R \cos \alpha_0 \sin \alpha_2$$

$$\begin{aligned}
 x_3 &= 0 \\
 x_4 &= R \sin \alpha_0 \\
 -\frac{\pi}{2} &< \alpha_0 < \frac{\pi}{2}, 0 \leq \alpha_2 < 2\pi
 \end{aligned}$$

is another *equator* of 4-dimensional sphere in *hyperplane*  $(x_1, x_2, x_4)$ . We can see this object og Figure 4.

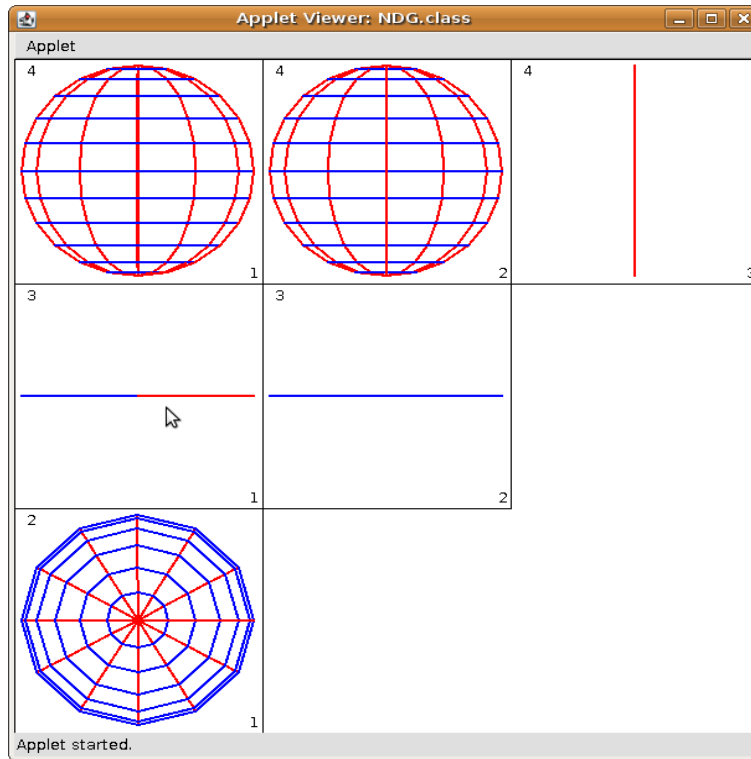


Figure 3: Second equator of 4D Sphere. Green lines are absent, since  $\alpha_1 = 0$ , red lines correspond to the change of  $\alpha_0$ , blue lines correspond to the change of  $\alpha_2$ , The equator is a 3-D sphere in hyperplane  $(x_1, x_2, x_4)$ . Coordinate  $x_3 = 0$

When  $\alpha_0 = \pm \frac{\pi}{2}$  we have poles

$$\begin{aligned}
 x_1 &= 0 \\
 x_2 &= 0
 \end{aligned}$$

$$\begin{aligned}
x_3 &= 0 \\
x_4 &= \pm R
\end{aligned}$$

Important to notice that in 3-D we have *axis of rotation* which comes through north and south poles, in 4-D we have *plane*  $(x_3, x_4)$ , around which we are able to rotate the sphere. Really, looking at Eq.(9-13) you can see that change of longitude  $\alpha_2$  has no effect on  $(x_3, x_4)$ . When longitude  $\alpha_2 = 0$  we get *zero meridian*:

$$x_1 = R \cos \alpha_0 \cos \alpha_1 \quad (15)$$

$$x_2 = 0 \quad (16)$$

$$x_3 = R \cos \alpha_0 \sin \alpha_1 \quad (17)$$

$$x_4 = R \sin \alpha_0 \quad (18)$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_1 < \frac{\pi}{2} \quad (19)$$

which is a 3-D half-sphere in subspace  $(x_1, x_3, x_4)$ .

Intersection of the zero meridian with first equator ( $\alpha_0 = \alpha_2 = 0$ ) is a circle

$$x_1 = R \cos \alpha_1, x_3 = R \sin \alpha_1, x_2 = x_4 = 0$$

in a plane  $(x_1, x_3)$ . Intersection of the zero meridian with second equator ( $\alpha_1 = \alpha_2 = 0$ ) is a circle

$$x_1 = R \cos \alpha_2, x_3 = R \sin \alpha_2, x_2 = x_4 = 0$$

in a plane  $(x_1, x_3)$ . In 3-D space it was a point.

This 3-D meridian can be build with the receipt from previous section, as two subsequent rotations of one point, slightly modifying Eq. (7):

$$\begin{vmatrix} R \cos \alpha_0 \cos \alpha_1 \\ 0 \\ R \cos \alpha_0 \sin \alpha_1 \\ R \sin \alpha_0 \end{vmatrix} = \begin{vmatrix} \cos \alpha_1, & 0, & -\sin \alpha_1, & 0 \\ 0, & 1, & 0, & 0 \\ \sin \alpha_1, & 0, & \cos \alpha_1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} \cos \alpha_0, & 0, & 0, & -\sin \alpha_0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ \sin \alpha_0, & 0, & 0, & \cos \alpha_0 \end{vmatrix} \begin{vmatrix} R \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad (20)$$

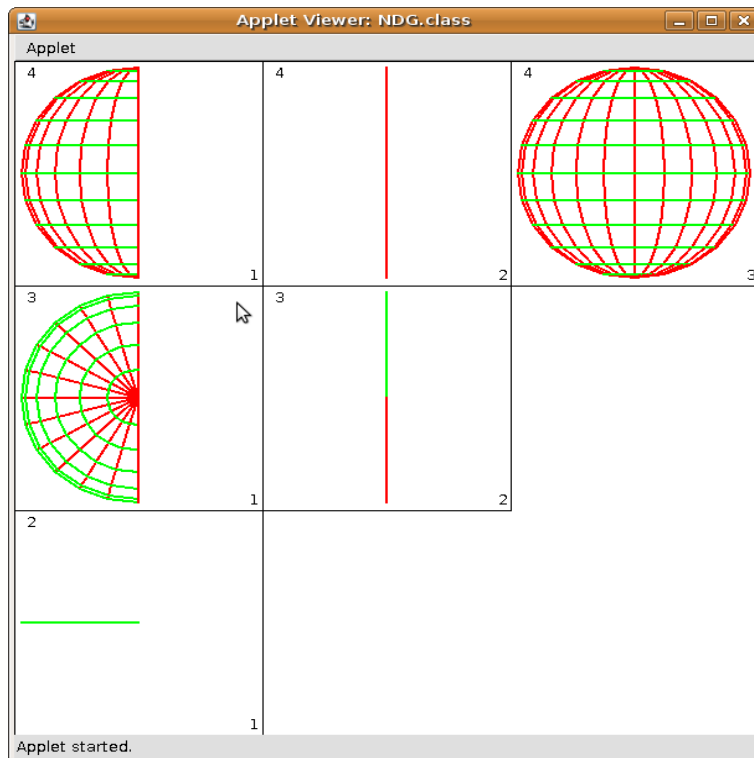


Figure 4: Zero meridian of 4D Sphere. Blue lines are absent, since  $\alpha_2 = 0$ , red lines correspond to the change of  $\alpha_0$ , green lines correspond to the change of  $\alpha_1$ . The meridian is a 3-D half-sphere in hyperplane  $(x_1, x_3, x_4)$ . Coordinate  $x_2 = 0$

First or rightmost rotation in Eq.8 is a rotation around the plane  $(x_2, x_3)$ , since those coordinates are not affected, and the leftmost is a rotation around the plane  $(x_2, x_4)$ . Similar to the 3-D case, if we *rotate this meridian around the plane  $(x_3, x_4)$*  we will get equation of 4-D sphere:

$$\begin{vmatrix} \cos \alpha_2, & -\sin \alpha_2, & 0, & 0 \\ \sin \alpha_2, & \cos \alpha_2, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} R \cos \alpha_0 \cos \alpha_1 \\ 0 \\ R \cos \alpha_0 \sin \alpha_1 \\ R \sin \alpha_0 \end{vmatrix} = \begin{vmatrix} R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \\ R \cos \alpha_0 \cos \alpha_1 \sin \alpha_2 \\ R \cos \alpha_0 \sin \alpha_1 \\ R \sin \alpha_0 \end{vmatrix} \quad (21)$$

Result:to receive a 3-D surface of 4-D sphere we must take a single point and perform 3 rotations, around 3 planes. Let us ascend at one more space dimension.

### 3 5-dimensional sphere

Equation of the *body* of the sphere in 4-D Euclidean space  $(x_1, x_2, x_3, x_4)$  is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq R^2 \quad (22)$$

It is a 5-D object. A *surface* of 5-D sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2 \quad (23)$$

is a 4-dimensional, since  $x_1, x_2, x_3, x_4, x_5$  are tied with equation (22), so only three of them are independent. It especially can be seen from parametric representation of sphere surface as

$$x_1 = R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \quad (24)$$

$$x_2 = R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \sin \alpha_3 \quad (25)$$

$$x_3 = R \cos \alpha_0 \cos \alpha_1 \sin \alpha_2 \quad (26)$$

$$x_4 = R \cos \alpha_0 \sin \alpha_1 \quad (27)$$

$$x_5 = R \sin \alpha_0 \quad (28)$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_1, \alpha_2 < \frac{\pi}{2}, 0 \leq \alpha_3 < 2\pi \quad (29)$$

where  $\alpha_0, \alpha_1, \alpha_2$  are *three latitudes* and  $\alpha_3$  is a *longitude*.

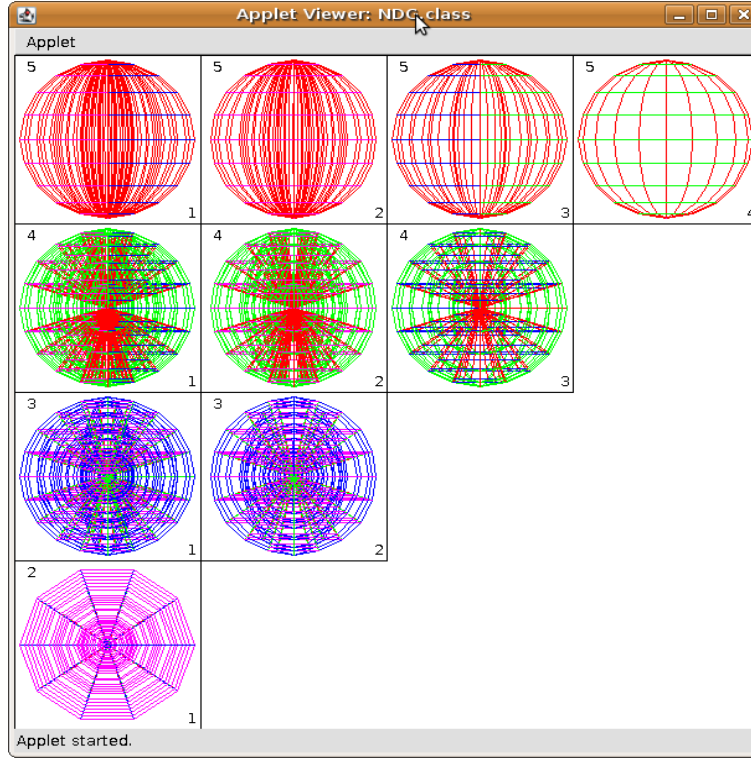


Figure 5: 5D Sphere. Red lines correspond to the change of  $\alpha_0$ , green lines correspond to the change of  $\alpha_1$ , blue lines correspond to the change of  $\alpha_2$ , magenta lines correspond to the change of  $\alpha_3$

If  $\alpha_0 = 0$  then

$$x_1 = R \cos \alpha_1 \cos \alpha_2 \cos \alpha_3$$

$$x_2 = R \cos \alpha_1 \cos \alpha_2 \sin \alpha_3$$

$$x_3 = R \cos \alpha_1 \sin \alpha_2$$

$$x_4 = R \sin \alpha_1$$

$$x_5 = 0$$

$$-\frac{\pi}{2} < \alpha_1, \alpha_2 < \frac{\pi}{2}, 0 \leq \alpha_3 < 2\pi$$

is a 3-dimensional surface of 4-dimensional sphere, first *equator*, of 5-dimensional sphere in *hyperplane*  $(x_1, x_2, x_3, x_4)$ .

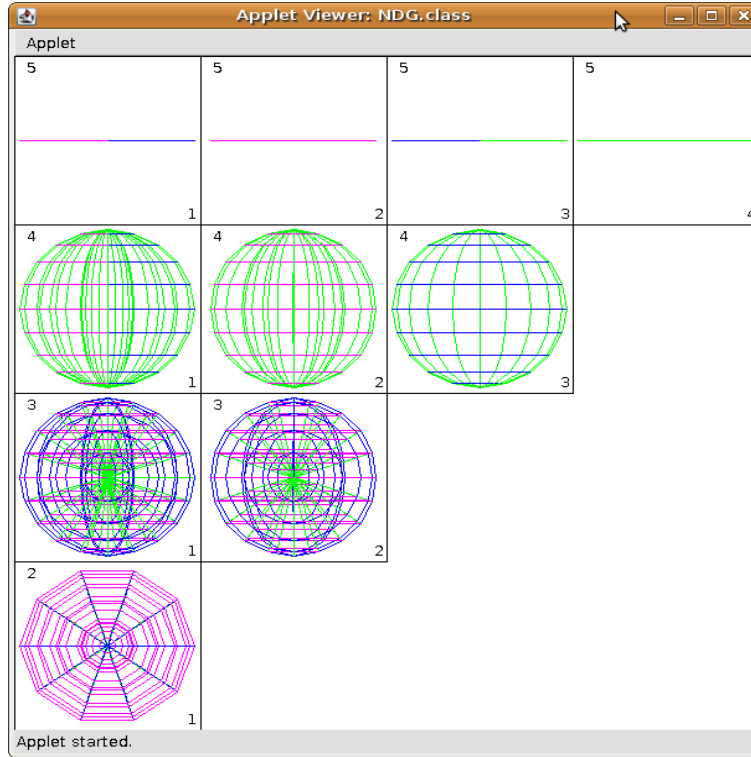


Figure 6: First equator of 5D Sphere. Red lines are absent, since  $\alpha_0 = 0$ , green lines correspond to the change of  $\alpha_1$ , blue lines correspond to the change of  $\alpha_2$ , magenta lines correspond to the change of  $\alpha_3$ . Equator is a 4-D sphere in hyperplane  $(x_1, x_2, x_3, x_4)$

If  $\alpha_1 = 0$  then

$$x_1 = R \cos \alpha_0 \cos \alpha_2 \cos \alpha_3$$

$$x_2 = R \cos \alpha_0 \cos \alpha_2 \sin \alpha_3$$

$$x_3 = R \cos \alpha_0 \sin \alpha_2$$

$$x_4 = 0$$

$$x_5 = R \sin \alpha_0$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_2 < \frac{\pi}{2}, 0 \leq \alpha_3 < 2\pi$$

is another *equator* of 5-dimensional sphere in *hyperplane*  $(x_1, x_2, x_3, x_5)$ .

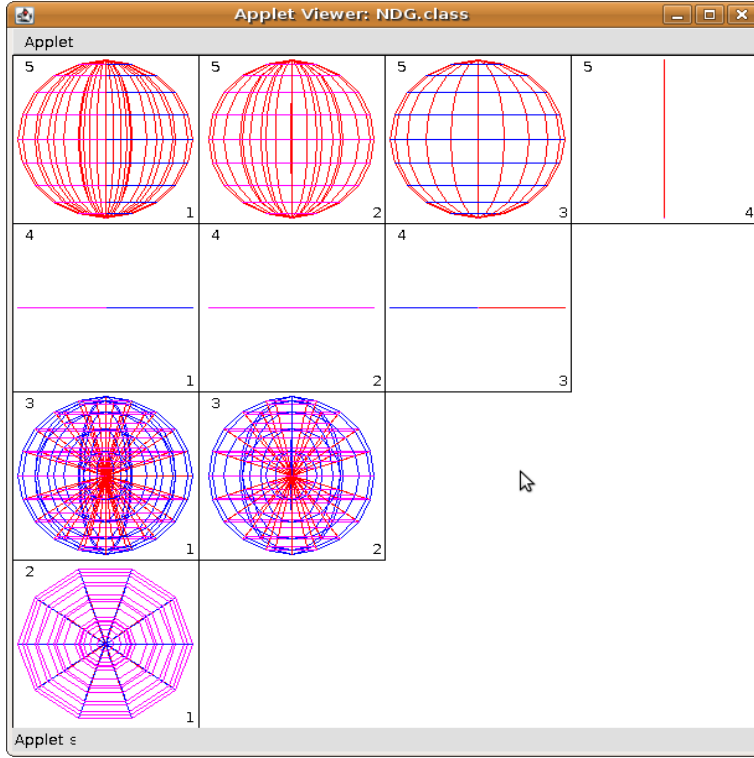


Figure 7: Second equator of 5D Sphere. Green lines are absent, since  $\alpha_1 = 0$ , red lines correspond to the change of  $\alpha_0$ , blue lines correspond to the change of  $\alpha_3$ , magenta lines correspond to the change of  $\alpha_3$ . Equator is a 4-D sphere in hyperplane  $(x_1, x_2, x_3, x_5)$

If  $\alpha_2 = 0$  then

$$x_1 = R \cos \alpha_0 \cos \alpha_1 \cos \alpha_3 \quad (30)$$

$$x_2 = R \cos \alpha_0 \cos \alpha_1 \sin \alpha_3 \quad (31)$$

$$x_3 = 0 \quad (32)$$

$$x_4 = R \cos \alpha_0 \sin \alpha_1 \quad (33)$$

$$x_5 = R \sin \alpha_0 \quad (34)$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_1, \alpha_2 < \frac{\pi}{2}, 0 \leq \alpha_3 < 2\pi \quad (35)$$

is another *equator* of 5-dimensional sphere in *hyperplane*  $(x_1, x_2, x_4, x_5)$ .

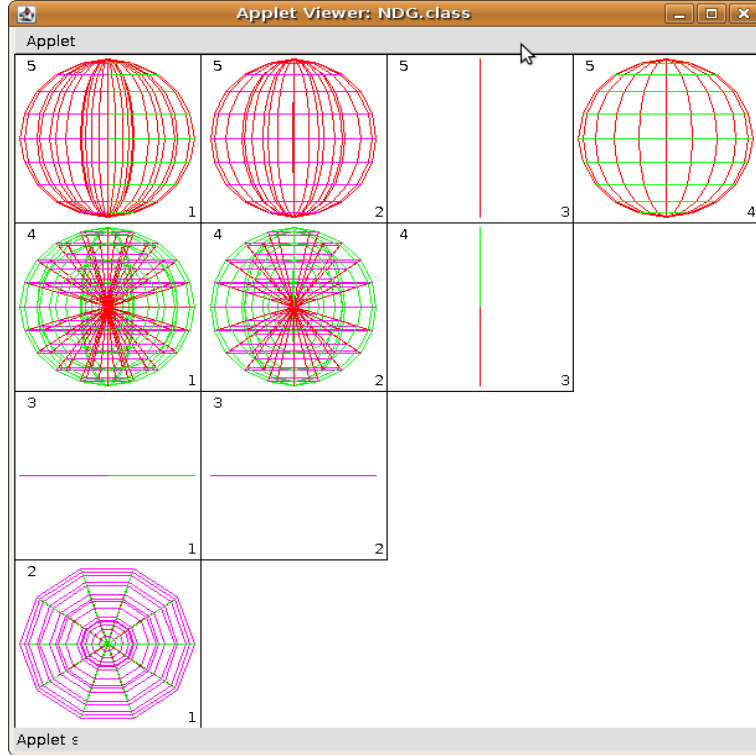


Figure 8: Third equator of 5D Sphere. Green lines are absent, since  $\alpha_1 = 0$ , red lines correspond to the change of  $\alpha_0$ , blue lines correspond to the change of  $\alpha_3$ , magenta lines correspond to the change of  $\alpha_3$ . Equator is a 4-D sphere in hyperplane  $(x_1, x_2, x_3, x_5)$

When  $\alpha_0 = \pm \frac{\pi}{2}$  we have couple of poles

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

$$\begin{aligned}
x_4 &= 0 \\
x_5 &= \pm R
\end{aligned}$$

When longitude  $\alpha_3 = 0$  we get *zero meridian*

$$x_1 = R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \quad (36)$$

$$x_2 = 0 \quad (37)$$

$$x_3 = R \cos \alpha_0 \cos \alpha_1 \sin \alpha_2 \quad (38)$$

$$x_4 = R \cos \alpha_0 \sin \alpha_1 \quad (39)$$

$$x_5 = R \sin \alpha_0 \quad (40)$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_1, \alpha_2 < \frac{\pi}{2} \quad (41)$$

which is a 4-D half sphere in subspace  $(x_1, x_3, x_4, x_5)$ .

Intersections of this meridian with equators are 3-D spheres, fact can be easily checked by the reader. Similar to the 4-D case, if we *rotate this meridian around the plane*  $(x_3, x_4, x_5)$  we will get equation of 5-D sphere:

$$\begin{vmatrix} \cos \alpha_3, & -\sin \alpha_3, & 0, & 0, & 0 \\ \sin \alpha_3, & \cos \alpha_3, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \\ 0 \\ R \cos \alpha_0 \cos \alpha_1 \sin \alpha_2 \\ R \cos \alpha_0 \sin \alpha_1 \\ R \sin \alpha_0 \end{vmatrix} = \begin{vmatrix} R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \\ R \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 \sin \alpha_3 \\ R \cos \alpha_0 \cos \alpha_1 \sin \alpha_2 \\ R \sin \alpha_0 \sin \alpha_1 \\ R \sin \alpha_0 \end{vmatrix} \quad (42)$$

And meridian itself can be provided by rotations of single point, just as it explained in 4-D sphere section.

Finally, answering the question in the article's title, we provide a table of some properties of the spheres in different space dimensions:

## 4 Conclusion

In this article we tried to demonstrate how analysis of elementary formulas unveils the richness of properties of a simple object like sphere, and how

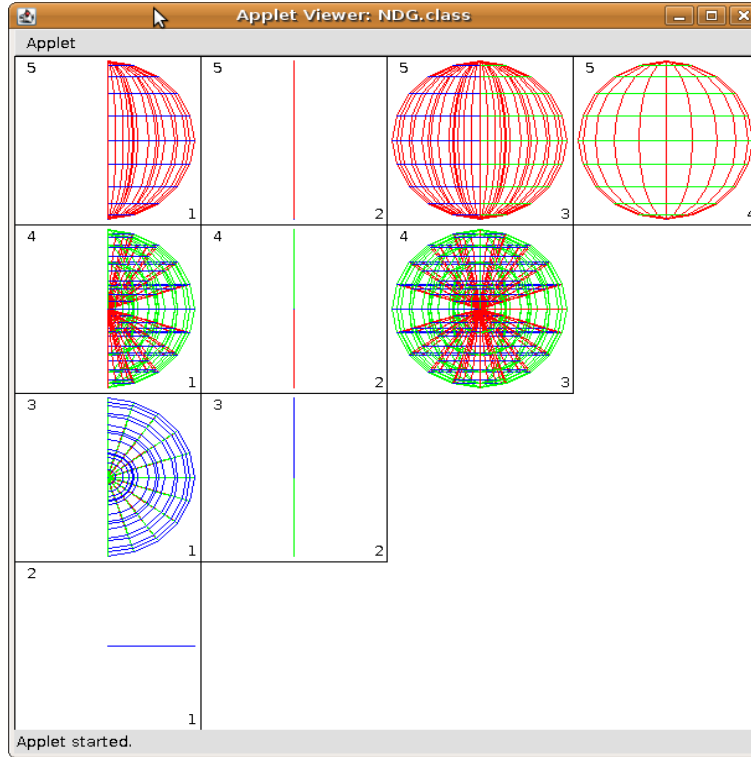


Figure 9: Zero meridian of 5D Sphere. Magenta lines are absent, since  $\alpha_3 = 0$ , Red lines correspond to the change of  $\alpha_0$ , green lines correspond to the change of  $\alpha_1$ , blue lines correspond to the change of  $\alpha_2$ . Meridian is a 4-D half sphere in hyperplane  $(x_1, x_3, x_4, x_5)$

	Equator	Meridian	Pole
3-D	1	1	2
4-D	2	1	2
5-D	3	1	2

Table 1: Elements of the sphere

those properties evolve with the raise of dimensions of space. We also produced a method of displaying of multidimensional object, which expands standard technical drafting. More examples of this approach reader can find at [www.asymptotus.com](http://www.asymptotus.com).

## References

- [1] G.A.Korn,T.M.Korn *Mathematical handbook for scientists and engineers*,Dover, NY, 2000